A Jurdjevic-Quinn-type theorem for stochastic nonlinear control systems.

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Abstract— **We consider stochastic nonaffine nonlinear control systems $x_t = x_0 + \int_0^t f(x_s,u)ds + \int_0^t g(x_s,u)d\omega_s$, (written in the sense of Itô), ω being a standard Wiener process, for which we give a sufficient condition for global stabilization by a bounded smooth state feedback which is explicitly given. This condition generalizes the well known Jurdjevic-Quinn result for deterministic affine control systems.**

Keywords— **Stochastic nonlinear control systems, stochastic stability, state feedback law, Lyapunov functions.**

I. Introduction

This paper deals with the question of stabilizability for stochastic nonlinear control differential equations written in the sense of Itô:

$$x_t = x_0 + \int_0^t f_0(x_s,u)ds + \sum_{j=1}^{p}\int_0^t f_j(x_s,u)d\omega_s^j, \qquad (1)$$

where $x_0 \in \mathbb{R}^n$, $u = (u_1,\ldots,u_m)^T$ is a R^m–valued control law, $\{\omega_t,\, t \geq 0\}$ is a standard $\mathbb{R}^p$–valued Wiener process defined on an usual probability space $(\Omega, \mathcal{F}, P)$, and $f_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $0 \leq j \leq p$, are smooth (C^∞) Lipschitz functions satisfying $f_j(0,0) = 0$ and there exists a positive constant K such that, for any $x \in \mathbb{R}^n$ and any $u \in \mathbb{R}^p$, $\sum_{j=0}^{p} \|f_j(x,u)\| \leq K(1 + \|x\| + \|u\|)$.

Stochastic control systems (1) are of interest for various reasons. As well known, a multitude of physical, engineering, biological, social, and managerial phenomena are either well approximated or reasonably modelled by control differential equations $dx_t/dt = f_t(x_t,u)$, for which many of the most basic questions concern stabilization around the equilibrium.

Again one often has situations where the coefficients, say $f_t(x_t,u)$, are not deterministic but of the random form $f_t(x_t,u) = b(x_t,u) + \sigma(x_t,u)\cdot$"noise", where b and σ are some given functions and where one does not know the exact behaviour of the noise term, but only its probability distribution. Of course in such a situation stochastic control differential equations (1) are more natural models than deterministic ones $dx/dt = f_0(x,u)$. For instance during these past decades there has been increasing effort to describe various facets of dynamic economic interactions with the help of stochastic differential processes. Traditional mathematical economics modelling focusses on transient and equilibrium interrelationships among production and consumption factors. stochastic differential processes provide a mechanism to incorporate the influences associated with randomness, uncertainties, and risk factors operating with respect to various economic units (stock prices, labour force, technology variables, etc.). Among other applications where stochastic differential equations occur to describe phenomena one can cite theoretical ecology and population genetics, and electrical dynamical systems. For the literature dealing with applications see e.g. [1], [3], [5], [6], [9], [11], [14], [15], [18].

For deterministic nonlinear control systems many techniques to study the stabilizability problem and to design stabilizing feedback laws are known. Historically, one of the first significant results is due to Jurdjevic and Quinn [7] who used the LaSalle's invariance principle to give a sufficient condition for the global stabilization of an affine nonlinear control system:

$$\dot{x} \triangleq \frac{dx}{dt} = X_0(x) + \sum_{i=1}^{m} u_i Y_0^i(x). \qquad (2)$$

with a linear (i.e. $X(x) = Ax$) and dissipative drift. Since then, various Jurdjevic-Quinn type sufficient conditions have been developed by several authors [8], [13], [16], [17]. In [16], it is proved that if there exists a positive definite and proper smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that: (i) the Lie derivative $X_0V(x) = \langle X_0(x), \nabla V(x)\rangle$ of V with respect to vector field X_0 satisfies $X_0V(x) \leq 0$, $\forall x \in \mathbb{R}^n$; (ii) the set $\{x \in \mathbb{R}^n | X_0^{k+1}V(x) = X_0^k Y_0^i V(x) = 0,\ k \in \mathbb{N},\ 1 \leq i \leq m\}$ is reduced to $\{0\}$; then the derivative of V along the trajectories of system (2) being given by $\dot{V}(x) = X_0V(x) + \sum_{i=1}^{m} u_i Y_0^i V(x)$, the smooth state feedback control law $u_i(x) = -Y_0^i V(x)$, $1 \leq i \leq m$, yields $\dot{V}(x) \leq 0$, that is to say a Lyapunov stable closed-loop system, and by application of LaSalle's invariance principle it stabilizes globally system (2).

In [4], Florchinger extends Jurdjevic-Quinn theorem to the particular class of stochastic affine control systems:

$$\begin{aligned} x_t \;=\;& x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^{m} u_i Y_0^i(x_s)\Big)ds \\ &+ \sum_{j=1}^{p}\int_0^t X_j(x_s)dw_s^j, \end{aligned} \qquad (3)$$

where only the drift term is corrupted by a noise. For these systems, the associated infinitesimal generator $\mathcal{L}$ satisfies $\mathcal{L}V(x) = L_0V(x) + \sum_{i=1}^{m} u_i Y_0^i V(x)$, $\forall x \in \mathbb{R}^n$, where L_0 is the second order differential operator defined by $L_0V(x) = X_0V(x) + (1/2)\sum_{j=1}^{p} \langle X_j(x), (\partial^2 V/\partial x^2)(x) X_j(x)\rangle$ and V is a given smooth positive definite and proper function. So, it follows that if: (i') $L_0V(x) \leq 0$ for all $x \in \mathbb{R}^n$; (ii') the set $\{x \in \mathbb{R}^n | L_0^{k+1}V(x) = L_0^k Y_0^i V(x) = 0,\ k \in \mathbb{N},\ 1 \leq i \leq m\}$ is reduced to $\{0\}$; then the smooth state feedback control law $u_i(x) = -Y_0^i V(x)$ yields $\mathcal{L}V(x) \leq 0$, which allows, as in [16] for the deterministic case, to state in [4], by application of the stochastic versions of Lyapunov theorem (see [10]) and LaSalle's invariance principle (see [12]), that the stochastic affine system (3) is globally asymptotically stabilizable in probability by the feedback law $u_i(x) = -Y_0^i V(x)$.

In order to illustrate the peculiar difficulty of the stochastic case in comparison with the deterministic one, which disappears for system (3), consider now affine control sys-

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tems of the form:

$$\begin{aligned} x_t &= x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^m u_i Y_0^i(x_s) \Big) ds \\ &\quad + \sum_{j=1}^{p_0} \int_0^t X_j(x_s) d\omega_{0\,s}^j \\ &\quad + \sum_{i=1}^m \sum_{j=1}^{p_i} \int_0^t u_i Y_j^i(x_s) d\omega_{i\,s}^j, \qquad (4) \end{aligned}$$

where ω_i, $0 \le i \le m$, is a standard $\mathbb{R}^{p_i}$-valued Wiener process such that ω_i and $\omega_{i'}$ are independent for $i \neq i'$. Contrary to system (3), for system (4) where every thing is corrupted by a noise, the associated infinitesimal generator $\mathcal{L}$, applied to a Lyapunov function V, leads to:

$$\begin{aligned} \mathcal{L}V(x) &= L_0 V(x) + \sum_{i=1}^m u_i Y_0^i V(x) \\ &\quad + \frac{1}{2} \sum_{i=1}^m u_i^2 \sum_{j=1}^{p_i} \langle Y_j^i(x), \frac{\partial^2 V}{\partial x^2}(x) Y_j^i(x) \rangle. \end{aligned}$$

So, it appears that the Jurdjevic-Quinn feedback $u_i(x) = -Y_0^i V(x)$, under conditions (i') and (ii'), is no more a stabilizing feedback for (4).

More generally for stochastic nonlinear systems of the form (1), for which the random parametric excitation depends on the control, $\mathcal{L}V(x)$ is a nonlinear expression on u which depends explicitly on the corrupted terms $f_j(x,u)$, $1 \le j \le p$. So, if one assumes that there exists a feedback $u(x)$ such that $\langle f_0(x,u(x)), \nabla V(x) \rangle \le 0$, the most difficult problem is now to prove the existence of a feedback law $\tilde{u}(x)$ yielding $\mathcal{L}V(x) \le 0$ and satisfying $\tilde{u}(x) = u(x)$ for $f_j = 0$, $1 \le j \le p$.

In [2], it is proved that Jurdjevic-Quinn type conditions (i') and (ii') remain sufficient for the stochastic affine control system (4) to be globally asymptotically feedback stabilizable in probability.

In this work, we give an extended version of these conditions under which the stochastic nonaffine control system (1) is globally asymptotically stabilizable in probability with a bounded smooth stabilizing feedback law $u = u(x)$ with $u(0) = 0$ and with an arbitrary choice of the bound. We make a constructive proof of this fact which provides an explicit design of bounded smooth stabilizing feedback laws.

it is worth while to point out that the above mentioned difficulty, that is peculiar to the stochastic case in comparison with the deterministic one, does not appear with the hypothesis that the coefficients associated with the noise in system (1) do not depend on the control, that is to say $\partial f_j / \partial u = 0$, $1 \le j \le p$. In this case the stochastic stabilization procedure is close to the deterministic one.

II. Notations and preliminaries

The first aim of this section is to recall some classical definitions and results on stability in probability of the zero solution of a stochastic differential equation (see e.g. [10]).

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be an usual probability space and $\{\omega_t, t \ge 0\}$ be a standard $\mathbb{R}^p$-valued Wiener process defined on this space. Denote by $(\mathcal{F}_t)_{t\ge 0}$ the complete right-continuous filtration generated by ω.

Let x_t be the $\mathbb{R}^n$-valued process solution of the stochastic differential equation written in the sense of Itô,

$$x_t = x_0 + \int_0^t X_0(x_s) ds + \sum_{k=1}^p \int_0^t X_k(x_s) d\omega_s^k, \qquad (5)$$

where $X_k(0) = 0$, $0 \le k \le p$. We assume that X_k are Lipschitz vector fields on $\mathbb{R}^n$ such that there exists a positive constant K such that, for any x in $\mathbb{R}^n$, $\sum_{k=0}^p \|X_k(x)\| \le K(1 + \|x\|)$. For any x_0 in $\mathbb{R}^n$, denote by $x_t(x_0)$, $t \ge 0$, the solution at time t of the stochastic differential equation (5) starting from the state x_0. Then, the different notions of stochastic stability that are used in this paper are the following.

Definition 1: The solution $x_t \equiv 0$ of the stochastic differential equation (5) is said to be stable in probability if for any $\epsilon > 0$, $\lim_{x_0 \to 0} P\left(\sup_{t>0} |x_t(x_0)| > \epsilon\right) = 0$. If, in addition, there exists a neighbourhood D of the origin such that $P\left(\lim_{t\to+\infty} |x_t(x_0)| = 0\right) = 1$, $\forall x_0 \in D$, the solution $x_t \equiv 0$ of the stochastic differential equation (5) is said to be asymptotically stable in probability. It is globally asymptotically stable in probability (GASP) if $D = \mathbb{R}^n$.

For $1 \le i,j \le n$, let $\mathcal{L}$ be the infinitesimal generator associated with the stochastic differential equation (5) defined for any function Ψ in $C^2(\mathbb{R}^n)$ by:

$$\begin{aligned} \mathcal{L}\Psi(x) &= \langle X_0(x), \nabla \Psi(x) \rangle \\ &\quad + \frac{1}{2} \sum_{k=1}^p \langle X_k(x), \frac{\partial^2 \Psi}{\partial x^2}(x) X_k(x) \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$.

Then, following criteria in terms of Lyapunov function for the stochastic stability hold (see [1],[10]).

Theorem 1: If there exist a neighbourhood D of the point $x = 0$ in $\mathbb{R}^n$, and a Lyapunov function V defined in D (i.e. a positive definite and proper function V mapping D into $\mathbb{R}$) such that $\mathcal{L}V(x) \le 0$ (resp. $\mathcal{L}V(x) < 0$), $\forall x \in D$, $x \neq 0$, then, the solution $x_t \equiv 0$ of the stochastic differential equation (5) is stable (resp. asymptotically stable) in probability. It is GASP if $\mathcal{L}V(x) < 0$, $\forall x \in \mathbb{R}^n$, $x \neq 0$.

By a *proper function* we mean a function $V : D \to \mathbb{R}$ such that $\{x \in \mathbb{R}^n \mid V(x) \le \xi\}$ is compact for each $\xi > 0$.

Recall also that if the solution $x_t \equiv 0$ of the stochastic differential equation (5) is stable in probability and there exists a Lyapunov function V defined in D such that $\mathcal{L}V(x) \le 0$, $\forall x \in D$, $x \neq 0$, then, the stochastic version of LaSalle's invariance principle (see [12]) allows to state that the stochastic process x_t converges in probability to the largest invariant set whose support is contained in the locus $\{x_t \mid \mathcal{L}V(x_t) = 0,\ \forall t \ge 0\}$.

Now, in order to state our results on the stabilization of stochastic control system (1), let us introduce the following notations and definitions.

Definition 2: The stochastic differential control system (1) will be said globally asymptotically feedback stabilizable in probability at the origin if there exists a feedback control law $u : \mathbb{R}^n \to \mathbb{R}^m$ with $u(0) = 0$ such that the zero solution $x_t \equiv 0$ of the closed-loop system $x_t = x_0 + \int_0^t f_0(x_s, u(x_s))ds + \sum_{j=1}^p \int_0^t f_j(x_s, u(x_s))d\omega_s^j$ is globally asymptotically stable in probability.

For $0 \le j \le p$ and $1 \le i \le m$, we associate with system (1) the vector fields X_j and Y_j^i defined by:

$$X_j(x) = f_j(x, 0), \quad Y_j^i(x) = \frac{\partial f_j}{\partial u_i}(x, 0), \tag{6}$$

and the second order differential operators L_0 and L_i defined for any function Ψ in $\mathcal{C}^2(\mathbb{R}^n)$ by:

$$\begin{aligned} L_0\Psi(x) &= \langle X_0(x), \nabla\Psi(x)\rangle \\ &\quad + \frac{1}{2}\sum_{j=1}^p \langle X_j(x), \frac{\partial^2\Psi}{\partial x^2}(x)X_j(x)\rangle, \end{aligned} \tag{7}$$

$$\begin{aligned} L_i\Psi(x) &= \langle Y_0^i(x), \nabla\Psi(x)\rangle \\ &\quad + \sum_{j=1}^p \langle X_j(x), \frac{\partial^2\Psi}{\partial x^2}(x)Y_j^i(x)\rangle. \end{aligned} \tag{8}$$

Definition 3: Stochastic nonlinear control system (1) is said to be a *Jurdjevic-Quinn type* stochastic system if there exists a positive definite and proper smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that: (h1) $LV(x) \le 0, \ \forall x \in \mathbb{R}^n$; (h2) the set $\{x \in \mathbb{R}^n | L_0^{k+1}V(x) = L_0^k L_i V(x) = 0, \ k \in \mathbb{N}, \ i = 1, \ldots, m\}$ is reduced to $\{0\}$.

Notice that for $f_j = 0$ (resp. $\partial f_j/\partial u = 0$), $1 \le j \le p$, conditions (h1) and (h2) reduce to (i) and (ii) (resp. (i') and (ii')).

Notice also that for the stochastic affine control system (4) that have been considered in [2], (h1) and (h2) reduce to (i') and (ii'). Indeed, set $p = \sum_{i=0}^m p_i$; $\tilde\omega = (\omega_0^1, \ldots, \omega_0^{p_0}, \ldots, \omega_m^1, \ldots, \omega_m^{p_m})$; for $1 \le i \le m$ and $1 \le j \le p$, $\tilde X_j = X_j$ if $1 \le j \le p_0$ and $\tilde X_j = 0$ otherwise ; $\tilde Y_j^i = Y_j^i$ if $\sum_{k=0}^{i-1} p_k + 1 \le j \le \sum_{k=0}^{i} p_k$ and $\tilde Y_j^i = 0$ otherwise. Then system (4) may be written on the form:

$$\begin{aligned} x_t &= x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^m u_i Y_0^i(x_s)\Big)ds \\ &\quad + \sum_{j=1}^p \int_0^t \Big(\tilde X_j(x_s) + \sum_{i=1}^m u_i \tilde Y_j^i(x_s)\Big)d\tilde\omega_s^j \end{aligned}$$

and one gets, for $1 \le i \le m$, $L_iV(x) = \big\langle\, Y_0^i(x), \nabla V(x)\,\big\rangle$, because of $\tilde X_j(x)\tilde Y_j^{i\,T}(x) = 0$, $1 \le j \le p$.

Finally, for a Lyapunov function V, associate with system (1) the smooth function $\psi_V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ defined by:

$$\begin{aligned} \psi_V(x, u) &= -g_0^T(x, u)\nabla V(x) \\ &\quad - \sum_{j=1}^p g_j^T(x, u)\frac{\partial^2 V}{\partial x^2}(x)X_j(x) \\ &\quad - \frac{1}{2}\sum_{j=1}^p g_j^T(x, u)\frac{\partial^2 V}{\partial x^2}(x)g_j(x, u)u, \end{aligned} \tag{9}$$

where

$$g_j(x, u) = \int_0^1 \frac{\partial f_j}{\partial u}(x, tu)dt, \quad 0 \le j \le p. \tag{10}$$

III. Fixed point stabilizability sufficient condition

The following proposition can now be stated as a preliminary result of stabilizability of Jurdjevic-Quinn type stochastic systems.

Proposition 1: Assume that system (1) is of Jurdjevic-Quinn type and let V be a Lyapunov function satisfying conditions (h1) and (h2). Assume also that, for any $x \in \mathbb{R}^n$, the function $\psi_V(x, \cdot)$ has a fixed point $u(x) = \psi_V(x, u(x))$ which is smooth and such that $u(0) = 0$. Then $u(x)$ is a globally stabilizing feedback for the stochastic system (1).

Proof: the functions f_j, $0 \le j \le p$, being smooth, one has from (6) and (10), $f_j(x, u) = X_j(x) + g_j(x, u)u$. Then denoting by $\mathcal{L}$ the infinitesimal generator associated with the closed-loop system

$$\begin{aligned} x_t &= x_0 + \int_0^t f_0(x_s, u(x_s))ds \\ &\quad + \sum_{j=1}^p \int_0^t f_j(x_s, u(x_s))d\omega_s^j, \end{aligned} \tag{11}$$

one has for all $x \in \mathbb{R}^n$:

$$\begin{aligned} \mathcal{L}V(x) &= \big\langle X_0(x) + g_0\big(x, u(x)\big)u(x), \nabla V(x)\big\rangle \\ &\quad + \frac{1}{2}\sum_{j=1}^p \Big\langle X_j(x) + g_j\big(x, u(x)\big)u(x), \\ &\quad \frac{\partial^2 V}{\partial x^2}(x)\left[X_j(x) + g_j\big(x, u(x)\big)u(x)\right]\Big\rangle, \end{aligned}$$

and by a simple computation one gets from (7), (8) and (9), $\mathcal{L}V(x) = L_0V(x) - \langle u(x), \psi_V(x, u(x))\rangle$. Hence, from $u(x) = \psi_V(x, u(x))$ and assumption (h1) one has $\mathcal{L}V(x) = L_0V(x) - \|u(x)\|^2 \le 0$, $\forall x \in \mathbb{R}^n$, and according with theorem 1, the zero solution $x_t \equiv 0$ of the closed-loop system (11) is stable in probability.

Besides, according to the stochastic version of LaSalle's invariance principle (see [12]), the stochastic process x_t converges in probability to the largest invariant set whose support is contained in the locus $\mathcal{L}V(x_t) = 0$ for all $t \ge 0$. Therefore, in order to prove that the zero solution of the closed-loop system is GASP it must be shown that for any complete solution x_t of (11) along which $\mathcal{L}V(x_t) = 0$ for all $t \ge 0$, one has necessarily $x_t = 0$ $\forall t \ge 0$.

Notice that, from (6) and (10), one has $g_j(x, 0) = \big(\, Y_j^1(x) \quad \ldots \quad Y_j^m(x)\,\big)$, and from (9) and (8) it follows that:

$$\begin{aligned} \psi_V(x, 0) &= -\big(\big\langle Y_0^1(x), \nabla V(x)\big\rangle, \ldots, \big\langle Y_0^m(x), \nabla V(x)\big\rangle\big)^T \\ &\quad - \sum_{j=1}^p \Big(\big\langle Y_j^1(x), \frac{\partial^2 V}{\partial x^2}(x)X_j(x)\big\rangle, \ldots \end{aligned}$$

$$\ldots, \left\langle Y_j^m(x), \frac{\partial^2 V}{\partial x^2}(x) X_j(x) \right\rangle \Big)^T$$
$$= \left(L_1 V(x), \ldots, L_m V(x)\right)^T .$$

Now $\mathcal{L}V(x) = 0$ if and only if $L_0 V(x) = 0$ and $u(x) = 0$, and since $u(x) = \psi_V(x, u(x))$, it turns out that $\mathcal{L}V(x) = 0 \Rightarrow L_0 V(x) = \cdots = L_m V(x) = 0$. So, for any complete solution x_t of the stochastic differential equation (11) for which $\mathcal{L}V(x_t) = 0$ for all $t \geq 0$, successive differentiations by means of Itô's formula yield $L_0^{k+1} V(x_t) = L_0^k L_i V(x_t) = 0$, for $t \geq 0$, $k \in \mathbb{N}$, and $i = 1, \ldots, m$. Hence, by assumption (h2), it follows that $x_t = 0$ for all $t \geq 0$, which completes the proof. ■

Remark 1: As an application of proposition 1, one can deduce the result of [2] on the stabilization of stochastic affine control system (4) provided that it is of Jurdjevic-Quinn type thanks to a Lyapunov function V satisfying conditions (i') and (ii'). Notice that for system (4) the function ψ_V defined in (9) satisfies $\psi_{V\,i}(x,u) = -Y_0^i V(x) - a_{V,i}(x) u_i$ where $a_{V,i}(x) = (1/2) \sum_{j=1}^{p_i} \langle Y_j^i(x), (\partial^2 V/\partial x^2)(x) Y_j^i(x) \rangle$, $1 \leq i \leq m$, and generally it has no smooth fixed point. Nevertheless, by using the static precompensator

$$u_i = \left(1 + a_{V,i}^2(x) - a_{V,i}(x)\right)^{-1/2} \tilde{u}_i, \quad 1 \leq i \leq m, \tag{12}$$

one transforms system (4) into

$$\begin{aligned} x_t &= x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^m \tilde{u}_i \tilde{Y}_0^i(x_s) \Big) ds \\ &\quad + \sum_{j=1}^{p_0} \int_0^t X_j(x_s) d\omega_{0\,s}^j \\ &\quad + \sum_{i=1}^m \sum_{j=1}^{p_i} \int_0^t \tilde{u}_i \tilde{Y}_j^i(x_s) d\omega_{i\,s}^j, \end{aligned} \tag{13}$$

where $\tilde{Y}_j^i = (1 + a_{V,i}^2 - a_{V,i})^{-1/2} Y_j^i$, $1 \leq i \leq m$, $0 \leq j \leq p_i$. Clearly (13) is of Jurdjevic-Quinn type in accordance with (4), and thanks to the same Lyapunov function V. Now, to system (13) one associates by (9) the function $\tilde{\psi}_V$ given by:

$$\begin{aligned} \tilde{\psi}_{V\,i}(x, \tilde{u}) &= -\tilde{Y}_0^i V(x) - \frac{u_i}{2} \sum_{j=1}^{p_i} \left\langle \tilde{Y}_j^i(x), \frac{\partial^2 V}{\partial x^2}(x) \tilde{Y}_j^i(x) \right\rangle \\ &= -\frac{Y_0^i V(x) + \dfrac{a_{V,i}(x)\tilde{u}_i}{\sqrt{1 + a_{V,i}^2(x) - a_{V,i}(x)}}}{\sqrt{1 + a_{V,i}^2(x) - a_{V,i}(x)}} \end{aligned}$$

which has a smooth fixed point $\tilde{u}(x)$ defined by $\tilde{u}_i(x) = -(1 + a_{V,i}^2(x) - a_{V,i}(x))^{1/2} (1 + a_{V,i}^2(x))^{-1} Y_0^i V(x)$. Therefore, following proposition 1, $\tilde{u}(x)$ stabilizes (13), and by (12) it yields the stabilizing feedback law $u_i(x) = -(1 + a_{V,i}^2(x))^{-1} Y_0^i V(x)$, for system (4). ■

IV. Stabilizability of Jurdjevic-Quinn type stochastic systems

Following Remark 1, the question now is what about the stabilizability of general Jurdjevic-Quinn type stochastic systems of the form (1). The next theorem will establish that any such a system is globally asymptotically stabilizable in probability by an arbitrarily bounded smooth feedback law.

Theorem 2: Assume that system (1) is of Jurdjevic-Quinn type and let V be a Lyapunov function satisfying conditions (h1) and (h2). Then, for any positive constant η, system (1) is globally asymptotically stabilizable in probability by means of a feedback law $u(x)$ satisfying $u(0) = 0$ and $\|u(x)\| \leq \eta$, $\forall\, x \in \mathbb{R}^n$.

Proof: Let us begin by the following remark on the function ψ_V associated with system (1) by (9), and that will be useful for the proof. If there exists a smooth function $k(x) > 0$ such that, for any $x \in \mathbb{R}^n$, the function $k(x)\psi_V(x, \cdot)$ has a fixed point $u(x) = k(x)\psi_V(x, u(x))$ which is smooth and such that $u(0) = 0$, then $u(x)$ is a globally stabilizing feedback for the stochastic system (1). As a matter of fact, the preliminary feedback $u = \sqrt{k(x)}\, \tilde{u}$ changes the original system (1) into the system

$$x_t = x_0 + \int_0^t \tilde{f}_0(x_s, \tilde{u}) ds + \sum_{j=1}^p \int_0^t \tilde{f}_j(x_s, \tilde{u}) d\omega_s^j, \tag{14}$$

where $\tilde{f}_j(x, \tilde{u}) = f_j(x, \sqrt{k(x)}\, \tilde{u})$, $0 \leq j \leq p$. One may also verify that the vector fields and the second order differential operators defined respectively by (6), (7) and (8) are changed into $\tilde{X}_j = X_j$, $\tilde{Y}_j^i = \sqrt{k}\, Y_j^i$, $\tilde{L}_0 = L_0$, and $\tilde{L}_i = \sqrt{k}\, L_i$. Therefore, (14) is a jurdjevic-Quinn type stochastic system in accordance with (1). Besides, if one denotes by ψ_V and $\tilde{\psi}_V$ the functions associated respectively with systems (1) and (14), a straightforward calculation shows that $\tilde{\psi}_V(x, \tilde{u}) = \sqrt{k(x)} \psi_V(x, \sqrt{k(x)}\, \tilde{u})$. So, if $u(x) = k(x)\psi_V(x, u(x))$, $u(0) = 0$, is a smooth fixed point of the function $k(x)\psi_V(x, \cdot)$, then one has: $\tilde{\psi}_V(x, u(x)/\sqrt{k(x)}) = \sqrt{k(x)} \psi_V(x, u(x)) = u(x)/\sqrt{k(x)}$. Hence, one can deduce from proposition 1 that $\tilde{u}(x) = u(x)/\sqrt{k(x)}$ stabilizes system (14), and accordingly $u(x)$ stabilizes system (1).

Now, for $\eta > 0$, let $K_1(x)$ and $K_2(x)$ be any smooth nonnegative real valued functions satisfying, for any $x \in \mathbb{R}^n$, $K_1(x) + K_2(x) \neq 0$, $K_1(x) \geq \sup_{\|u\| \leq \eta} \|\psi_V(x, u)\|$ and $K_2(x) \geq \sup_{\|u\| \leq \eta} \|(\partial \psi_V/\partial u)(x, u)\|$. Let $\alpha : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the smooth function defined by $\alpha(x, u) = \eta (K_1(x) + 2\eta K_2(x))^{-1} \psi_V(x, u)$. Then, for any $x \in \mathbb{R}^n$ and any $u \in \mathbb{R}^m$ such that $\|u\| \leq \eta$, one has $\|\alpha(x, u)\| \leq \eta$ and $\|(\partial \alpha/\partial u)(x, u)\| \leq 1/2$. So, on the one hand, applying the fixed point theorem one can deduce that there exists a unique continuous function $\theta : \mathbb{R}^n \to \mathbb{R}^m$, with $\theta(0) = 0$, satisfying for all $x \in \mathbb{R}^n$, $\|\theta(x)\| \leq \eta$ and $\alpha(x, \theta(x)) = \theta(x)$. On the other hand, the implicit function theorem applies to the function $\gamma(x, u) = \alpha(x, u) - u$ in each $x_0 \in \mathbb{R}^n$ since $\gamma(x_0, \theta(x_0)) = 0$ and the jacobian matrix $(\partial \gamma/\partial u)(x_0, \theta(x_0)) = (\partial \alpha/\partial u)(x_0, \theta(x_0)) - I_m$ is invertible. So, there exist a neighbourhood $\mathcal{V} \times \mathcal{U}$ of $(x_0, \theta(x_0))$

and $v : \mathcal{V} \to \mathcal{U}$ such that $v(x_0) = \theta(x_0)$ and $\gamma(x, v(x)) = 0$, $\forall x \in \mathcal{V}$. Now $v \in \mathcal{C}^\infty(\mathcal{V}, \mathcal{U})$ because of γ is $\mathcal{C}^\infty$, but the equation $\gamma(x, u) = 0$ has a unique solution $\theta(x)$ defined on $\mathbb{R}^n$, and so, $\theta_{|\mathcal{V}} = v$ and then θ is $\mathcal{C}^\infty$.

By setting $k(x) = \eta(K_1(x) + 2\eta K_2(x))^{-1}$, it turns out from the remark in the beginning of the proof that $u(x) = \theta(x)$ is a globally asymptotically stabilizing feedback for system (1), and the proof is completed. ■

In order to illustrate the feasibility of theorem 2, let us apply it to a stochastic affine control system of the form:

$$\begin{aligned} x_t &= x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^m u_i Y_0^i(x_s) \Big) ds \\ &\quad + \sum_{j=1}^p \int_0^t \Big(X_j(x_s) + \sum_{i=1}^m u_i Y_j^i(x_s) \Big) d\omega_s^j \quad (15) \end{aligned}$$

which is assumed to be of Jurdjevic-Quinn type thanks to a Lyapunov function V. Since $f_j(x, u) = X_j(x) + \sum_{i=1}^m u_i Y_j^i(x)$, one gets by a simple computation from (9) and (10) $\psi_V(x, u) = -h_V(x) - H_V(x)u$ where $h_V(x) = (L_1V(x), \ldots, L_mV(x))^T$ and $H_V(x)$ is the $m \times m$ matrix whose (i, k)th entry is $(1/2)\sum_{j=1}^p \langle Y_j^i(x), (\partial^2 V/\partial x^2)(x) Y_j^k(x)\rangle$. Hence, for a fixed $\eta > 0$, by taking $k(x) = \eta(K_1(x) + 2\eta K_2(x))^{-1}$ where $K_1(x)$ and $K_2(x)$ are smooth functions such that $K_1(x) + K_2(x)$ does not vanish on $\mathbb{R}^n$, $K_1(x) \geq \sup_{\|u\| \leq \eta} \|h_V(x) + H_V(x)u\|$ and $K_2(x) \geq \|H_V(x)\|$, one gets $k(x)\psi_V(x, u) = -k(x)h_V(x) - k(x)H_V(x)$. Now, for all $x \in \mathbb{R}^n$, the $m \times m$ matrix $k(x)H_V(x)$ satisfies $\|k(x)H_V(x)\| \leq 1/2$, and so, the matrix $I_m + k(x)H_V(x)$ is invertible. Therefore, the fixed point $u(x)$ of the function $k(x)\psi_V(x, \cdot)$, which satisfies $u(0) = 0$ and $\|u(x)\| \leq \eta$, can actually be explicitly computed: $u(x) = -(I_m + k(x)H_V(x))^{-1} k(x)h_V(x)$, and it is a globally asymptotically stabilizing feedback law for system (15).

In particular, for Jurdjevic-Quinn type stochastic affine control systems of the form (4), one has $h_V(x) = \big(Y_0^1V(x), \ldots, Y_0^mV(x)\big)^T$ and $H_V(x) = \mathrm{diag}\,(a_{V,1}(x), \ldots, a_{V,m}(x))$. Thus, the above procedure yields bounded feedback laws of the form $u_i(x) = -\eta(K_1(x) + 2\eta K_2(x))^{-1} Y_0^iV(x)$, where a possible choice of K_1 and K_2 is given by $K_1(x) = 1 + \sum_{i=1}^m \big(Y_0^iV(x)\big)^2 + \eta\sum_{i=1}^m \big(1 + a_{V,i}^2(x)\big)$ and $K_2(x) = \sum_{i=1}^m \big(1 + a_{V,i}^2(x)\big)$.

V. Explicit design of stabilizing feedback

Notice that, as established above, theorem 2 gives an existential stabilizability result in the sense that, even if for particular cases as Jurdjevic-Quinn type stochastic affine control systems of the form (15) the fixed point can be exactly computed, it does not yield, in general, explicitly the stabilizing feedback control law. By providing an explicit design of such a feedback, the next theorem is more close to practical preoccupations in automatic control.

the functions f_j, $1 \leq j \leq p$, being smooth, recall that, from the Taylor expansion formula, one has:

$$f_j(x, u) = f_j(x, 0) + \frac{\partial f_j}{\partial u}(x, 0)u + \tilde{f}_j(x, u, u), \quad (16)$$

where $\tilde{f}_j : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$, is defined by

$$\tilde{f}_j(x, v, w) = \int_0^1 (1 - t)\frac{\partial^2 f_j}{\partial u^2}(x, tv)(w, w)dt.$$

The notation $(\partial^2 f_j/\partial u^2)(x, tv)$ is used for the second order derivative of f_j with respect to u at (x, tv), that is to say the bilinear application from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^n$ defined by $(\partial^2 f_j/\partial u^2)(x, tv)(w, \tilde{w}) = (w^T(\partial^2 f_j^1/\partial u^2)(x, tv)\tilde{w}, \ldots, w^T(\partial^2 f_j^n/\partial u^2)(x, tv)\tilde{w})^T$ with $f_j^1, \ldots, f_j^n$ the component functions of f_j.

Besides, for a Lyapunov function V, let $\varphi_V : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the smooth function defined by:

$$\begin{aligned} \varphi_V(x, v, w) &= \langle \tilde{f}_0(x, v, w), \nabla V(x) \rangle \\ &\quad + \frac{1}{2}\sum_{j=1}^p \mathrm{Tr}\Big(A_j(x, v, w)\, \frac{\partial^2 V}{\partial x^2}(x) \Big) \quad (17) \end{aligned}$$

where the $n \times n$ matrix $A_j(x, v, w)$ is defined by:

$$\begin{aligned} A_j(x, v, w) &= X_j(x)\tilde{f}_j^T(x, v, w) + \tilde{f}_j(x, v, w)X_j^T(x) \\ &\quad + \sum_{i=1}^m v_i \Big[Y_j^i(x)\tilde{f}_j^T(x, v, w) + \tilde{f}_j(x, v, w){Y_j^i}^T(x) \Big] \\ &\quad + \sum_{i_1, i_2 = 1}^m w_{i_1} w_{i_2} Y_j^{i_1}(x){Y_j^{i_2}}^T(x) + \tilde{f}_j(x, v, v)\tilde{f}_j^T(x, v, w). \end{aligned}$$

Notice that the real valued function φ_V is homogeneous of degree 2 with respect to w. Then one can state:

Theorem 3: Assume that system (1) is of Jurdjevic-Quinn type and let V be a Lyapunov function satisfying conditions (h1) and (h2). Then for any $\eta > 0$ and any smooth functions $K_1(x)$ and $K_2(x)$ satisfying, $\forall x \in \mathbb{R}^n$, $K_1(x) + K_2(x) \neq 0$ and

$$K_1(x) \geq \sup_{\|v\| \leq \eta, \|w\| = 1} |\varphi_V(x, v, w)|, \quad (18)$$

$$K_2(x) \geq \big\| \big(L_1V(x), .., L_mV(x)\big) \big\|, \quad (19)$$

the stochastic control system (1) is globally stabilizable by means of the feedback:

$$u(x) = \frac{-\eta}{\eta K_1(x) + K_2(x)}\big(L_1V(x), .., L_mV(x)\big)^T, \quad (20)$$

which satisfies $\|u(x)\| \leq \eta$, $\forall x \in \mathbb{R}^n$.

Proof: The inequality $\|u(x)\| \leq \eta$ is an immediate consequence of (19), and (20). Moreover, from (6), and (16) the closed-loop system (1-20) is given by the stochastic differential equation:

$$\begin{aligned} x_t &= x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^m u^i(x_s)Y_0^i(x_s) \\ &\qquad + \tilde{f}_0\big(x_s, u(x_s), u(x_s)\big) \Big) ds \end{aligned}$$

$$+\sum_{j=1}^{p}\int_{0}^{t}\Big(X_j(x_s)+\sum_{i=1}^{m}u^i(x_s)Y_j^i(x_s)$$
$$+\tilde{f}_j\big(x_s,u(x_s),u(x_s)\big)\Big)d\omega_s^j \qquad (21)$$

Then, denoting by $\mathcal{L}$ the infinitesimal generator associated with the stochastic differential equation (21), one has:

$$\begin{aligned}\mathcal{L}V(x) \;=\; & \Big\langle X_0(x_s)+\sum_{i=1}^{m}u^i(x_s)Y_0^i(x_s) \\ & \qquad +\tilde{f}\big(x_s,u(x_s),u(x_s)\big),\nabla V(x)\Big\rangle \\ & +\frac{1}{2}\sum_{j=1}^{p}\Big\langle\Big[X_j(x_s)+\sum_{i=1}^{m}u^i(x_s)Y_j^i(x_s) \\ & \qquad +\tilde{f}_j\big(x_s,u(x_s),u(x_s)\big)\Big], \\ & \frac{\partial^2 V}{\partial x^2}(x)\Big[X_j(x_s)+\sum_{i=1}^{m}u^i(x_s)Y_j^i(x_s) \\ & \qquad +\tilde{f}_j\big(x_s,u(x_s),u(x_s)\big)\Big]\Big\rangle\end{aligned}$$

and by a simple computation one gets from (7), (8) and (17), $\mathcal{L}V(x) = L_0V(x) + \sum_{i=1}^{m}u^i(x)L_iV(x) + \varphi_V(x,u(x),u(x))$. It follows that, for $x\in\mathbb{R}^n$ such that $u(x)=0$ one has $\mathcal{L}V(x)=L_0V(x)$, and otherwise, from (20) and the homogeneity property of $\varphi_V(x,v,w)$ with respect to w one gets:

$$\mathcal{L}V(x)=L_0V(x)-\frac{\|u(x)\|^2\left[1-K(x)\,\varphi_V\Big(x,u(x),\frac{u(x)}{\|u(x)\|}\Big)\right]}{K(x)}$$

where $K(x)=\eta(\eta K_1(x)+K_2(x))^{-1}$. Besides, one has $1-K(x)\,\varphi_V(x,u(x),u(x)/\|u(x)\|)\geq 0$ because of (18) and $\|u(x)\|\leq\eta$, and so one gets from assumption (h1), $\mathcal{L}V(x)\leq 0$, $\forall\, x\in\mathbb{R}^n$. Hence, according with theorem 1, the zero solution $x_t\equiv 0$ of the stochastic differential equation (21) is stable in probability. Moreover, it follows from (19), (19) and (20) that if $u(x)\neq 0$ then $K_2(x)\neq 0$ and so $1-K(x)\,\varphi_V\,(x,u(x),u(x)/\|u(x)\|)\neq 0$, and, from (20), it turns out that $\mathcal{L}V(x)=0$ if and only if $L_iV(x)=0,\ i=1,\dots,m$. Therefore, the proof can be continued, thanks to the stochastic version of LaSalle's invariance principle, exactly as in the proof of Proposition1.■

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